PRIFYSGOL ABERYSTWYTH UNIVERSITY

EXAMINATION FOR ENTRANCE SCHOLARSHIPS JANUARY 2017

MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches.
- Answer all questions in Section A and two questions from Section B.
- Calculators are permitted, provided they are silent, self-powered, without communication facilities, and incapable of holding text or other material that could be used to give a candidate an unfair advantage. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
- Statistical tables will be provided.

Section A

1. Simplify the following expressions:

(a) $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} - \sqrt{15}$;

(b) $\log_{10} 2 + \log_{10} 5$;

(c) $\dfrac{x-3}{x^2-2x-3} + \dfrac{2}{x^2-1}$.

[13 marks]

2. Differentiate the following expressions with respect to x, simplifying your answer as far as possible:

(a) $2x^2 + e^x$;

(b) $\sqrt{1+x^2}$;

(c) $2\sin^2(5x^2) + \cos(10x^2)$.

[12 marks]

3. Integrate the following expressions with respect to x:

(a) e^{-2x};

(b) xe^{-2x};

(c) xe^{-2x^2}.

[8 marks]

4. The curve C has equation $y = x^2 - 4x - 5$.

(a) Write C in the following two forms: $y = (x-a)^2 + b$ and $y = (x-p_1)(x-p_2)$, where a, b, p_1 and p_2 are constants to be determined.

(b) Find the point at which the symmetry axis of C intersects C. Find the two points at which C intersects the x-axis.

(c) Sketch the graph of C on the xy-plane, indicating the three points found in (b).

[14 marks]

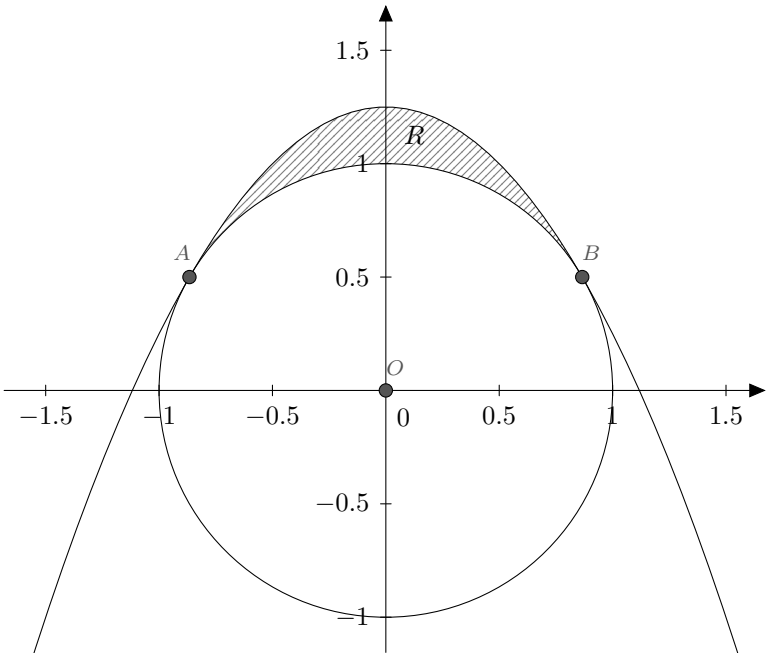
5. Determine the value of a such that $x = 3$ is a solution of the equation

$$x^3 - 6x^2 + 11x = a.$$

Factorise the resulting polynomial $x^3 - 6x^2 + 11x - a$ completely into its linear factors. [13 marks]

Section B

6. A value of a is chosen so that the circle $x^2+y^2 = 1$ lies inside the parabola $x^2+y = a$, touching it at exactly two points A and B, as shown in the figure below.



(a) Find the coordinates of A and B and show that $a = \frac{5}{4}$. [7 marks]

(b) Determine the angles and the area of the triangle OAB, and the area of the minor sector of the circle between the line segments OA and OB. [6 marks]

(c) Determine the area of the shaded region R bounded below by the circle and above by the parabola, between the points A and B. [7 marks]

7. (a) A car is being driven along a road at a steady 12 m/s when the driver suddenly notices that there is a fallen tree blocking the road 40 m ahead. The driver immediately applies the brakes giving the car a constant deceleration of 3 m/s^2.

(i) How far in front of the tree does the car come to rest? [5 marks]

(ii) If the driver had not reacted immediately and the brakes were applied 1.75 seconds later, with what speed would the car have hit the tree? [5 marks]

(b) Two particles, A and B, are able to move along the same straight line on a smooth horizontal surface. The particles A and B have masses m kg and 0.2 kg respectively. Two experiments are conducted. In the first experiment, A moves with velocity 5 m/s and B moves with velocity v m/s before colliding to form a single particle C, which moves with velocity 2 m/s. In the second experiment, the initial velocity of A and the initial speed of B are the same as before, but the direction of motion of B is reversed. The particles again collide and form a single particle C, which now moves with velocity 3 m/s.

(i) For each experiment, state, in terms of m and v, the total momentum of the particles before they collide. [4 marks]

(ii) Find the mass of particle A and the initial speed of particle B. [6 marks]

8. (a) Fence panels are supplied in heights of 180 cm. The actual length, X cm, of a panel may be modelled by a Normal distribution with mean 182 cm and standard deviation 1.6 cm.

Find the probability that a randomly selected panel has a height

(i) less than 180 cm. [2 marks]

(ii) more than 183 cm. [2 marks]

(iii) between 181 cm and 183 cm. [2 marks]

(b) The mass of an electrical component, Y, may be modelled by a Normal distribution with mean μ and standard deviation σ, i.e., $Y \sim N(\mu, \sigma^2)$.

An analysis of a batch of components showed that

$$P(Y < 4.5) = 0.2.$$

(i) Find the value of z such that

$$4.5 - \mu = z \times \sigma,$$

where z is a value of $Z \sim N(0, 1)$. [2 marks]

(ii) If you also know that

$$P(Y > 4) = 0.975,$$

find μ and σ. [4 marks]

(c) If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent, then for any real numbers a and b,

$$aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2).$$

Suppose that the lengths of screws can be modelled by a Normal distribution with mean 5.1 cm and standard deviation 0.03 cm. You may assume that the lengths of different screws are independent.

Suppose that two screws are selected at random.

(i) What is the probability that the sum of the lengths of the two screws is greater than 10.16 cm? [3 marks]

(ii) What is the probability that the difference between the two lengths is less than 0.01 cm?
(Hint: Use the fact that

$$P(|X| < K) = P(-K < X < K).)$$

[5 marks]