PRIFYSGOL ABERYSTWYTH UNIVERSITY

ENTRANCE EXAMINATION
MARCH 2022

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers (including any diagrams, graphs or sketches) should be written on paper, and scanned into a **single** PDF file. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.

Information and relevant formulas

- For motion in a circle of radius r, the transverse velocity is ωr, and the magnitude of the radial acceleration is $\omega^2 r$, where ω is the angular speed.
- Statistical tables will be provided. Note that the tables refer to the **right-hand** tails of the distributions, that is, probabilities of the form $p = \mathbb{P}(X \geq x)$ where X is a random variable and x an **upper** percentage point of its distribution.
- If X is a discrete random variable with values $x_1, \ldots, x_n$, then the mean and variance of X are given by

$$\mathrm{E}(X) = \sum_{i=1}^{n} x_i \, \mathrm{P}(X = x_i), \qquad \mathrm{Var}(X) = \sum_{i=1}^{n} x_i^2 \, \mathrm{P}(X = x_i) - \mathrm{E}(X)^2.$$

- Formulas related to standard distributions (e.g. for probability, mean, and variance) can be found on the back page of the statistical tables.

Section A

1. Simplify the following expressions as far as possible, showing your workings.

(a) $\dfrac{34}{5i+3} - \dfrac{11-7i}{2+i}$ [4 marks]

(b) $\left|(3\mathbf{i} - 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + \mathbf{k})\right|$ [3 marks]

(c) $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$ [4 marks]

2. Solve the equation $2z + 7\overline{z} = 5 - 3i - z$, for the complex number z. (Here $\overline{z}$ is the complex conjugate of z.) [6 marks]

3. Explain why the matrix $\begin{pmatrix} 3 & -2 \\ -9 & 6 \end{pmatrix}$ has no inverse. [2 marks]

4. Consider the complex numbers $z = \frac{1}{2}(\sqrt{3}i - 1)$ and $w = z^2$.

(a) Express w in the form $r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi \leq \theta \leq \pi$. [6 marks]

(b) Show that $w^2 = z$. [2 marks]

(c) Find an integer n for which $z^n = 1$, explaining your reasoning. [2 marks]

5. The complex number z satisfying $|z - 3i| = |z + 1 - i|$ is represented by the point $P(x, y)$ in an Argand diagram. Find the equation of the locus of P in terms of x and y, and interpret it geometrically. [6 marks]

6. Consider the quartic equation $x^4 + 4x^3 + 11x^2 + 34x + 130 = 0$.

(a) Let $z = 2i - 3$. Simplify the expressions of z^2, z^3, and z^4, and hence show that z is a root of the above equation. [4 marks]

(b) Find the other three roots, explaining your method for each root. [8 marks]

7. Prove by mathematical induction that

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 2^n & 3^n - 2^n \\ 0 & 3^n \end{pmatrix},$$

for all positive integers n. [7 marks]

8. A quadratic polynomial $f(x)$ has real coefficients and $f''(0) = -2$. Given that the equation $f(x) = 0$ has a non-real root z, find expressions for $f(0)$ and $f'(0)$ in terms of z and $\overline{z}$. [6 marks]

Section B

9. Consider the points $A(1,1,1)$, $B(2,0,3)$, $C(5,1,2)$, and $D(4,2,0)$. Let Π be the plane containing A, B, and C.

(a) Find (in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) the vectors $\mathbf{AB}$, $\mathbf{BC}$, $\mathbf{CD}$, and $\mathbf{DA}$. [2 marks]

(b) Find the equation of the plane Π in the form $\mathbf{r} \cdot \mathbf{n} = 1$, and hence show that the point D lies in Π as well. [8 marks]

(c) Show that $ABCD$ is a rectangle on the plane Π. [2 marks]

(d) Find the vector equation of the line L perpendicular to the plane Π and intersecting it at the centre of the rectangle $ABCD$. [3 marks]

(e) Find the coordinates of a point P which lies on the line L determined in part (d) such that the triangle APB is equilateral. [5 marks]

10. An object of mass m moves in a circle on a smooth horizontal surface, constrained by a light inextensible string of length l attached to a point at height h above the surface, as shown in the figure. The angular speed ω of the object remains constant throughout the motion, and the string is taut.

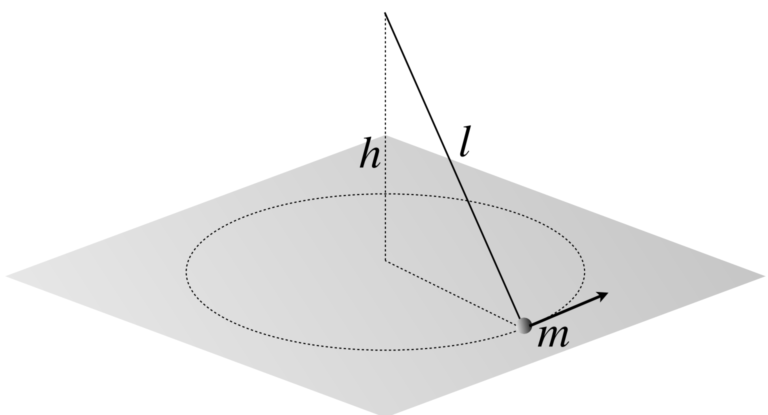
(a) Sketch a figure showing all the forces acting on the object, as well as the horizontal and vertical components of the tension force in the string. [4 marks]

(b) Show that the vertical component of the tension has magnitude $m\omega^2 h$. [7 marks]

(c) In the remaining calculations, let $h = 20\,\text{cm}$, and use the value $g = 9.8\,\text{ms}^{-2}$ for the gravitational acceleration.

(i) If the reaction force exerted on the object by the surface is 20% of the weight of the object (in magnitude), find the angular speed ω. [5 marks]

(ii) Find the largest value of ω for which the object remains on the surface. [4 marks]



11. (a) The discrete random variable X has the following probability distribution:

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.2	0.3	0.2	a	b

Here, a and b are positive constants.

(i) Show that $a + b = 0.3$. [2 marks]

(ii) Given that $\mathrm{E}(X) = 2.8$, obtain a second equation involving a and b. Hence determine the value of a and the value of b. [3 marks]

(iii) Given that X_1 and X_2 are independent observations of the random variable X, determine $\mathrm{P}(X_1 + X_2 \leq 4)$, explaining your reasoning. [4 marks]

(b) Let N represent the number of buses passing a bus stop in one hour. We can model N as a Poisson random variable with rate parameter $\lambda = 1.5$.

(i) Write down the mean and variance of N. [2 marks]

(ii) If $Y = 3N - 4$, what are the mean and variance of Y? [2 marks]

(iii) Using the appropriate formula for probabilities of the form $\mathrm{P}(N = k)$, calculate the probability that 3 or fewer buses are observed in a particular hour. Show your working clearly. [3 marks]

(iv) If N_1, N_2 and N_3 are the numbers of buses passing the stop between 9.00am and 10.00am, between 10.00am and 11.00am and between 11.00am and 12.00pm respectively, what is the distribution of $M = N_1 + N_2 + N_3$, the number of buses passing between 9.00am and 12.00pm? [2 marks]

(v) With M as in (iv), find $\mathrm{P}(M \geq 2)$, using either an appropriate formula or a probability table. Indicate clearly which method you used. [2 marks]